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Quantum kinematics and geometric quantization ★

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Abstract

Quantum kinematics on a configuration manifold (Angermann et al., 1983; Tolar 1991) extends the notion of schrdinger systems (Segal, 1960; Šťovček, 1981). Geometric quantization sets as its goal the construction of quantum objects using the geometry of the corresponding classical objects as a point of departure (Kirillov, 1992; Koodhouse, 1992). In this paper, we prove that differential quantum kinematics on a smooth manifold Q derive from the geometric quantization on the cotangent bundle T^*Q.

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1. Introduction

Let Q be a differential manifold. The family Γ_Q of non-relativistic quantum mechanical systems, localized and moving on Q, is characterized by a set $\mathcal{K}_Q$ of "kinematical objects" called *Borel kinematics* on Q. $\mathcal{K}_Q$ is quantized [1] by Angermann and Doebner and Tolar [2] via a mapping of $\mathcal{K}_Q$ into the set of self-adjoint operators in some Hilbert space H, such that those properties of $\mathcal{K}_Q$ survive, which are both characteristic for Γ_Q and can be used for a rigorous mathematical formulation.

On the other hand, the cotangent bundle $M = T^*Q$ of Q with the canonical 2-form

$$\omega = \mathrm{d}p_a \wedge \mathrm{d}q^a$$

is a symplectic manifold, where the q's are coordinates on Q and the p's are the corresponding components of covectors. The Souriau–Kostant formula [5] gives prequantizations of

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(M, ω). By the real polarization which has the cotangent spaces as its leaves, a number of quantizations of (M, ω) can be obtained. In this paper, we prove that these quantizations are almost the same as the differential quantum kinematics given by [1].

2. Quantum kinematics on smooth manifolds

For a differential manifold Q, $\mathcal{K}_Q = (\mathcal{L}(Q), \mathcal{X}_c(Q))$ is called *Borel kinematics* on Q, where $\mathcal{L}(Q)$ is the σ-algebra of Borel sets of Q—position observables, and $\mathcal{X}_c(Q) = \{X \mid X$ is a smooth complete vector field on $Q\}$ —momentum observables.

Definition 2.1. A triple $(H, \mathbf{E}, \mathbf{P})$ is called *a quantum Borel kinematics* on Q iff
(1) H is a separable Hilbert space;
(2) $\mathbf{E}$ is an elementary spectral measure on $\mathcal{L}(Q)$ in H;
(3) $\mathbf{P} : \mathcal{X}_c(Q) \to SA(H)$ (the set of self-adjoint operators on H) is a map with the following properties:
   (a) $\mathbf{P}(X)$ is the infinitesimal generator of a unitary one-parameter group of "shifts" along X for all $X \in \mathcal{X}_c(Q)$,
   (b) $\mathbf{P}$ is local,
   (c) $\mathbf{P}$ is a partial Lie homomorphism; and the domain v^{∞} (see [1]) is dense in H.

Here all the notations are the same as Definition 2 in [1] except that (c) means

$$\mathbf{P}(X + aY) = \mathbf{P}(X) + a\mathbf{P}(Y)$$

for $X, Y \in \mathcal{X}_c(Q), a \in \mathbb{R}$ whenever $X + aY \in \mathcal{X}_c(Q)$, and

$$[\mathbf{P}(X), \mathbf{P}(Y)] = -i\hbar\mathbf{P}([X, Y])$$

for all $X, Y \in \mathcal{X}_c(Q)$ whenever $[X, Y] \in \mathcal{X}_c(Q)$, where $2\pi\hbar$ is Plank's constant.

Remark. Here quantum kinematics are defined as in [1]. In fact, they are so-called quantum Borel 1-kinematics (QBK^1) in [2] where $\mathbf{E}$ is only a projection-valued measure in the definition of quantum kinematics.

Two quantum Borel kinematics $(H_j, \mathbf{E}_j, \mathbf{P}_j)$, $j = 1, 2$, on Q are called *equivalent* iff there exists a unitary map $\phi : H_1 \to H_2$ such that

$$\phi\mathbf{E}_1(B)\phi^{-1} = \mathbf{E}_2(B), \qquad \phi\mathbf{P}_1(X)\phi^{-1} = \mathbf{P}_2(X)$$

for all $B \in \mathcal{L}(Q)$, $X \in \mathcal{X}_c(Q)$.

There is a natural correspondence between the set $C(Q, \mathbb{C})$ of complex-valued functions on Q and the set $\Gamma(L_0)$ of sections of the fibration $L_0 = (Q \times \mathbb{C}, pr_1, Q)$ with pr_1 being the natural projection of $M \times \mathbb{C}$ onto M:

$$s \leftrightarrow f_s, \quad s \in \Gamma(L_0), \quad f_s \in C(Q, \mathbb{C}) \text{ such that } s(q) = (q, f_s(q)), q \in Q.$$

It is easy to show that f_s is Borel-measurable iff s is Borel-measurable (with respect to the σ-algebra $\mathcal{L}(Q) \otimes \mathcal{L}(\mathbb{C})$ on $Q \times \mathbb{C}$).

Fix a smooth Borel measure γ on Q. Define

$$\langle\ ,\ \rangle_0 : pr_1^{-1}(q) \times pr_1^{-1}(q) \to \mathbb{C}, \quad \langle (q,z),(q,z')\rangle_0 = z\bar{z}', \quad q \in Q,$$

$$L^2(L_0,\gamma) = \left\{ s \in \Gamma(L_0) | s \text{ is measurable}, \int_Q \langle s,s\rangle_0 \, \mathrm{d}\gamma < +\infty \right\}.$$

Following [1], up to unitary equivalence, any quantum Borel kinematics $(H, \mathbf{E}, \mathbf{P})$ on Q has the form:

(1) $H = L^2(L_0,\gamma)$,

(2) $\mathbf{E}(B)\psi = \xi_B\psi\ \forall B \in \mathcal{L}(Q)$, $\psi \in H$, where ξ_B denotes the indicator function of B.

A quantum Borel kinematics $(H, \mathbf{E}, \mathbf{P})$ on Q is called *differential* if there exists a differential structure $D = (\tau, a)$ on the point set $Q \times \mathbb{C}$, where τ is a Hausdorff topology for $Q \times \mathbb{C}$ and a is a maximal C^∞-altas of charts compatible with τ, such that $L_D = ((Q \times \mathbb{C}, D), pr_1, Q, \mathbb{C})$ is a complex line bundle over Q with hermitian metric $\langle\ ,\ \rangle_0$, $\mathcal{L}(Q \times \mathbb{C}, \tau) = \mathcal{L}(Q) \otimes \mathcal{L}(\mathbb{C})$ and the domain v^∞ equals to the complex vector space $\Gamma_0^\infty(L_D)$ of compactly supported differential sections of L_D which is dense in $L^2(L_0,\gamma)$ [1, Theorem 4].

Theorem 2.1. *Let Q be a differential manifold and γ be a fixed smooth Borel measure on Q:*

(1) *For every pair (L, r) consisting of a complex line bundle over Q with the hermitian metric $\langle\ ,\ \rangle$ and hermitian linear connection ∇ with vanishing curvature, and a real number r,*

$$H_L = L^2(L, \langle\ ,\ \rangle, \gamma), \qquad \mathbf{E}(B)\psi = \xi_B\psi \quad \text{for } B \in \mathcal{L}(Q),$$

$$\mathbf{P}(X)|\Gamma_0^\infty(L) = -\mathrm{i}h\,\nabla_X - \left(\tfrac{1}{2}i + r\right) h\, div_\gamma X \quad \forall X \in \mathcal{X}_c(Q)$$

define a differential quantum Borel kinematics $(H, \mathbf{E}, \mathbf{P})$ on Q.

(2) *Every differential quantum Borel kinematics on Q is equivalent to one given by* (1).

(3) *Two differential quantum Borel kinematics $(H_j, \mathbf{E}_j, \mathbf{P}_j)$ in* (1) *characterized by (L_j, r_j), $j = 1, 2$, are equivalent if and only if $r_1 = r_2$ and if there is an isometric isomorphism of L_1 onto L_2, which transforms the connections into each other. Therefore, the set of equivalence classes of differential quantum Borel kinematics on Q can be mapped bijectively onto $\pi_1(Q)^* \times \mathbb{R}$ where $\pi_1(Q)$ denotes the fundamental group of Q and $\pi_1(Q)^*$ its group of characters.*

Remark.

(1) The property of flatness of ∇ in Theorem 2.1 derives from

$$[\mathbf{P}(X), \mathbf{P}(Y)] = -\mathrm{i}h\mathbf{P}([X,Y])$$

for all $X, Y \in \mathcal{X}_c(Q)$ whenever $[X, Y] \in \mathcal{X}_c(Q)$.

(2) There are similar results for quantum Borel r-kinematics ($r > 1$) of type 0 [2].

For the proof, see Theorems 6 and 7 in [1].

We note that a differential quantum Borel kinematics $(H, \mathbf{E}, \mathbf{P})$ given by (L, r) in Theorem 2.1 automatically induces maps:

$$Q : C^\infty(Q, \mathbb{R}) \to End(H) \quad \text{and} \quad \mathbf{P} : \mathcal{X}(Q) \to End(H)$$

such that

$$Q(f)\psi = f\psi, \qquad \mathbf{P}(X)\psi = (-\mathrm{i}h \nabla_X - (\tfrac{1}{2}\mathrm{i} + r)h\, div_\gamma\, X)\psi$$

for $f \in C^\infty(Q, \mathbb{R})$, $X \in \mathcal{X}(Q)$, $\psi \in v^\infty = \Gamma_0^\infty(L)$, where the set $End(H)$ of operators of the Hilbert space H is a Lie algebra with the bracket $[\ ,\ ]_h$:

$$[A, B]_h = \mathrm{i}h^{-1}(AB - BA), \quad A, B \in End(H),$$

and $\mathcal{X}(Q)$ is the set of vector fields on Q. Meanwhile, let

$$g_k = C^\infty(Q, \mathbb{R}) \oplus \mathcal{X}(Q).$$

Define $[\ ,\ ] : g_k \times g_k \to g_k$ as follows:

$$[X + f, Y + g] = [X, Y] + (Xg - Yf) \quad \forall X, Y \in \mathcal{X}(Q), \quad f, g \in C^\infty(Q, \mathbb{R}),$$

then g_k is a Lie algebra. According to [1], the differential quantum Borel kinematics given by (L, r) induces a Lie homomorphism from g_k into $End(H)$, denote it by $\pi^k_{(L,\mathrm{i}/2+r)}$. In fact, replacing $\frac{1}{2}\mathrm{i} + r$ by any $c \in \mathbb{C}$ in the formulas of $\mathbf{P}$, we also get a Lie homomorphism. Denote it by $\pi^k_{(L,c)}$.

Definition 2.2. A *h-representation* of a Lie algebra g is a pair (H, π), where H is a Hilbert space and

$$\pi : g \to End(H)$$

is a Lie homomorphism. Two h-representations (H_j, π_j), $j = 1, 2$, of g are called *equivalent* if there exists a unitary isomorphism $\phi : H_1 \to H_2$ such that $\pi_1(X)\phi = \phi\pi_2(X)$ for all $X \in g$.

Theorem 2.2. *For any $c \in \mathbb{C}$ and any complex line bundle L with the hermitian metric $\langle\ ,\ \rangle$ and the hermitian linear connection ∇ with vanishing curvature,*

$$\pi^k_{(L,c)} : g_k \to End(H_L)$$

is a h-representations of g_k, where

$$\pi^k_{(L,c)}(f + X)\psi = (-\mathrm{i}h \nabla_X - ch\, div_\gamma X + f)\psi$$

for $f \in C^\infty(Q, \mathbb{R})$, $X \in \mathcal{X}(Q)$, $\psi \in v^\infty = \Gamma_0^\infty(L)$. In particular, all the differential quantum Borel Kinematics $\pi^k_{(L,\mathrm{i}/2+r)}$ on Q are h-representations of g_k.

3. Geometric quantizations of T^*Q

Let $M = T^*Q$ be the cotangent bundle of a smooth manifold Q. Then $\omega = \mathrm{d}\theta$ is a symplectic form on T^*Q, where $\theta = p_a\,\mathrm{d}q^a$ is the canonical one-form, the q's are local coordinates on Q, and the p's are the corresponding components of covectors. A Poisson bracket is defined in $C^\infty(M, \mathbb{R})$:

$$\{F, G\} = \sum_a \left(\frac{\partial F}{\partial p_a}\frac{\partial G}{\partial q^a} - \frac{\partial F}{\partial q^a}\frac{\partial G}{\partial p_a} \right), \quad F, G \in C^\infty(M, \mathbb{R}).$$

Following Dirac, a *prequantization* is a linear mapping $F \to \check{F}$ of the Poisson algebra $C^\infty(M, \mathbb{R})$ into $SA(H)$ for some Hilbert space H, having the properties:

(1) $\check{1} = 1$;

(2) $[F_1, F_2]\check{} = [\check{F}_1, \check{F}_2]_h$.

Theorem 3.1 (Kostant [5]). *For every complex line bundle B over M with a hermitian metric and a hermitian linear connection with the curvature $h^{-1}\omega$, H = closure of $\Gamma_0^\infty(B)$ and $\check{F} = -ih\,\nabla_{X_F} + F$ define a prequantization, here X_F is the Hamilton vector field of F. Two prequantizations given by B_j, $j = 1, 2$, are equivalent (i.e. equivalent as h-representations of the Lie algebra $C^\infty(M, \mathbb{R})$) if and only if there is an isometric isomorphism of B_1 onto B_2, which transforms the connections into each other. So the set of equivalence classes of presentations of $M = T^*Q$ can be parametrized by $\pi_1(M)^* \cong \pi_1(Q)^*$.*

Now we choose the vertical polarization P of $M = T^*Q$: $P_m = T_m(T^*_{pr(m)}Q)$, $m \in M$, where $pr : M \to Q$ is the natural projective map of bundle T^*Q on Q. Then $Q = M/P$. For simplicity, assume that Q is oriented.

Denote by $\Delta_Q \to Q$ the line bundle $\Lambda^n_{\mathbb{C}}(Q)$, where $n = \dim Q$, $\Omega^p_{\mathbb{C}}(M)$ the space of complex p-forms on M, and

$$V_P(M) = \{X \in \mathcal{X}(M) | X_m \in P_m, m \in M\}.$$

Then for $\beta \in \Omega^n_{\mathbb{C}}(M)$, β is the pull-back of a section of Δ_Q iff $X \vdash \beta = 0$ and $X \vdash \mathrm{d}\beta = 0$ for all $X \in V_P(M)$, and the Lie derivative $L_Z\beta$ is also the pull-back of a section of Δ_Q if $Z \in \mathcal{X}(M)$ whose flow preserves P, where $\vdash$ denotes the contraction of X with β.

Let $K_P \subset \Lambda^n_{\mathbb{C}}(M)$ be the canonical line bundle whose fibre at $m \in M$ is the one-dimensional subspace of $\Lambda^n T^*_{m,\mathbb{C}}M$ of forms α such that $X \vdash \alpha = 0$ for every $X \in P_m$. It is obvious that $K_P = pr^*\Delta_Q$. Since Q is oriented, the transition functions of Δ_Q and K_P can all be made real and positive. So we can take their square roots $\sqrt{\Delta_Q}$ and $\sqrt{K_P}$ by taking the square roots of the transition functions.

The covariant derivative ∇_X on K_P is defined for $X \in V_P(M)$ by $\nabla_X\beta = X \vdash \mathrm{d}\beta$. The sections of K_P which are covariantly constant along P are the pull-backs of n-forms on Q. If Z is a vector field on M whose flow preserves P, then the Lie derivative L_Z maps sections of K_P to sections of K_P. The ∇_X and L_Z can pass to the bundle $\delta_P = \sqrt{K_P}$ where they are determined by

$$2(\nabla_X \tau)\tau = \nabla_X \tau^2, \qquad 2(L_Z \tau)\tau = L_Z \tau^2.$$

Here τ is a section of δ_P.

Let B be a prequantum bundle, that is, a complex line bundle over M with a hermitian metric $\langle\ ,\ \rangle$ and a hermitian linear connection ∇ with the curvature $h^{-1}\omega$. Set $B_P = B \otimes \delta_P$. Define

$$V_B = \{\tilde{s} = s\tau \in \Gamma^\infty(B_P) |\ \nabla_X \tilde{s} = (\nabla_X s)\tau + s \nabla_X \tau = 0\}.$$

If $\tilde{s} = s\tau$ and $\tilde{s}' = s'\tau' \in V_B$, then $\langle \tilde{s}, \tilde{s}' \rangle = \langle s, s' \rangle \tau\tau' \in \Gamma^\infty(K_P)$ and for $X \in V_P(M)$,

$$\nabla_X \langle \tilde{s}, \tilde{s}' \rangle = \langle \nabla_X \tilde{s}, \tilde{s}' \rangle + \langle \tilde{s}, \nabla_X \tilde{s}' \rangle = 0.$$

Hence we can identify $\langle \tilde{s}, \tilde{s}' \rangle$ with an n-form on Q and define an inner product on V_B by

$$(\tilde{s}, \tilde{s}') = \int_Q \langle \tilde{s}, \tilde{s}' \rangle.$$

The completion of $\{\tilde{s} \in V_B | (\tilde{s}, \tilde{s}) < \infty\}$ is a Hilbert space H_B.

Let $\{q^a\}$ be local coordinates on Q, and $\{p_a\}$ the corresponding components of covectors, then a classical observable that generates a flow preserving P is locally of the form

$$F(q, p) = v^a(q) p_a + u(q),$$

where v's and u are smooth real-valued functions of q's. It is easy to see that

$$g_P = \{F \in C^\infty(M, \mathbb{R}) |\ \text{whose flow preserving } P\}$$

is a Lie subalgebra of the Poisson algebra $C^\infty(M, \mathbb{R})$. For $F \in g_P$, the possible choice of the corresponding quantum observable is the operator $\hat{F}$ that acts on V_B by

$$\hat{F}\tilde{s} = \check{F}(s)\tau + c s L_{X_F} \tau,$$

where $\tilde{s} = s\tau \in V_B$, $c \in \mathbb{C}$. It is easy to check that $\hat{F}$ is well-defined iff $c = -\mathrm{i}h$. So

$$\hat{F}\tilde{s} = \check{F}(s)\tau - \mathrm{i}h s L_{X_F} \tau,$$

where $\tilde{s} = s\tau \in V_B$.

Theorem 3.2. [2] *The Hilbert space H_B and the mapping $F \to \hat{F}$, $F \in g_P$, define a geometric quantization of M which is a h-representation of the Lie algebra g_P.*

4. Quantum kinematics and geometric quantization

Now suppose Q, Δ_Q, B as above. Since Q is oriented, $\sqrt{\Delta_Q}$ is trivial. Fix a non-vanishing section τ' of $\sqrt{\Delta_Q}$. Then $\gamma = \tau'^2$ is a section of Δ_Q which can be considered as a Borel measure on Q. Set $\tau = pr^*\tau' \in \Gamma(\delta_P)$. We have $\nabla_X \tau = 0$ for every $X \in V_P(M)$, and

$$V_B = \{\tilde{s} = s\tau | s \in \Gamma^\infty(B), \nabla_X s = 0, X \in V_P(M)\}.$$

For $F \in g_P$ and $c \in \mathbb{C}$, define

$$\tilde{F}(s\tau) = \check{F}(s)\tau + cL_{X_F}\tau, \quad s\tau \in V_B.$$

Theorem 4.1. *$\pi^g_{(B,c)} : F \to \tilde{F}$, $F \in g_P$, defines a h-representation of g_P.*

For the quantum objects g_K in quantum kinematics and g_P in geometric quantizations, we have the following conclusion.

Theorem 4.2. *As Lie algebras, g_K is isomorphic to g_P.*

Proof. For $f \in C^\infty(Q, \mathbb{R})$, $X \in \mathcal{X}(Q)$, define

$$\phi(f + X)(q, \alpha) = f(q) + \alpha(X_q),$$

where $q \in Q, \alpha \in T^*_q Q$. Then $\phi(f + X) \in g_P$. It is easy to check that $\phi : g_K \to g_P$ is a Lie isomorphism. □

Let $B \to M$ be a complex line bundle with a hermitian metric $\langle\ ,\ \rangle$ and a hermitian linear connection ∇ with the curvature $h^{-1}\omega = h^{-1}\,\mathrm{d}\theta$, where $\theta = p_a\,\mathrm{d}q^a$ is the canonical one-form on $M = T^*Q$. Find a collection $\{(U_j, \tau_j)\}$ of local trivializations of B such that $\{U_j\}$ is a contractible open cover of M, $\tau_j : U_j \times \mathbb{C} \cong \pi^{-1}(U_j)$ and $\nabla_X s_j = 0$ for any $X \in V_P(M)$, where $s_j = \tau_j(\cdot, 1)$ is the unit section of (U_j, τ_j). Then there exist $c_{jk} \in C^\infty(U_j \cap U_k, \mathbb{C})$, and $\theta_j \in \Omega(U_j)$ such that

$$s_k = c_{jk}s_j, \qquad \nabla s_j = -\mathrm{i}h^{-1}\theta_j s_j, \qquad \theta_k - \theta_j = \mathrm{i}h^{-1}(\mathrm{d}c_{jk}/c_{jk}).$$

Since the curvature of ∇ is $h^{-1}\omega = h^{-1}\,\mathrm{d}\theta$, $\theta_j = \theta + \mathrm{d}g_j$ for some $g_j \in C^\infty(U_j)$.

Now it is easy to see that $\{V_j = pr\,U_j\}$ and $\{\mathrm{d}_{jk} = pr\,c_{jk}\}$ determine a complex line bundle $L_B \to Q$ which is the restriction of B on Q and has the hermitian metric $\langle\ ,\ \rangle|Q$. Moreover, $\{\alpha_j = \theta_j|V_j = \mathrm{d}g_j|V_j\}$ determines a hermitian linear connection ∇ of L_B with the vanishing curvature. We can check that

$$Xg_j = 0, \quad Xc_{jk} = 0 \qquad \text{for any } X \in V_P(M).$$

So $c_{jk} = pr^*\,\mathrm{d}_{jk}$ and $\mathrm{d}g_j = pr^*\alpha_j$.

It is easy to prove the following result.

Theorem 4.3. *$\tilde{\rho} : B \to L_B$ is a bijection between complex line bundles over M with a hermitian metric and a hermitian linear connection with the curvature $h^{-1}\omega = h^{-1}\,\mathrm{d}\theta$ and complex line bundles over Q with a hermitian metric and a hermitian linear connection with the vanishing curvature. It induces a unitary isomorphism*

$$\rho : H_B \to L^2(L_B, \nu), \quad \rho(s\tau) = s|Q.$$

Let $\phi : g_K \to g_P$ be the isomorphism in Theorem 4.2. Choose $F \in g_P$, that is, in the cannonical coordinate system on M,

$$F(p, q) = v^a(q)p_a + u(q)$$

such that $Y \in \mathcal{X}(Q)$ when we define $Y_q = v^a(q)(\partial/\partial q^a)$ for $q \in Q$, and $u \in C^\infty(Q)$, $F = \phi(Y) + \phi(u)$. The Hamilton vector field of F is

$$X_F = Y - \left(\frac{\partial v^a}{\partial q^b} p_a + \frac{\partial u}{\partial q^b}\right) \frac{\partial}{\partial p_b}.$$

Let $B \to M$ be a complex line bundle with a hermitian metric $\langle\ ,\ \rangle$ and a hermitian linear connection ∇ with the curvature $h^{-1}\omega = h^{-1}\,\mathrm{d}\theta$, (U_0, τ_0) be a local trivialization of B such that $\nabla_X s_0 = 0$ where s_0 is the unit section of (U_0, τ_0) and $X \in V_P(M)$. We have $\nabla s_0 = -\mathrm{i}h^{-1}(\theta + \mathrm{d}g)s_0$ for some $g \in C^\infty(U_0)$ and $Xg = 0$ for any $X \in V_P(M)$. Now choose $\tilde{s} = s\tau \in V_B \subset H_B$, then $s|U_0 = fs_0$ for some $f \in C^\infty(U_0)$, $Xf = 0 \forall X \in V_P(M)$, and for $c \in \mathbb{C}$,

$$\begin{aligned}
\pi^g_{(B,c)}(F)\tilde{s} &= \check{F}(s)\tau + \tfrac{1}{2}chsL_{X_F}\tau \\
&= (-ih\,\nabla_{X_F} s + Fs)\tau + \tfrac{1}{2}chsL_{X_F}\tau \\
&= [-ih(X_F f - \mathrm{i}h^{-1}\theta(X_F) - \mathrm{i}h^{-1}X_F g)s_0 + Ffs_0]\tau + \tfrac{1}{2}chsL_{X_F}\tau \\
&= [-ih(Yf - \mathrm{i}h^{-1}\,\mathrm{d}g(Y))s_0 + ufs_0]\tau + \tfrac{1}{2}chsL_{X_F}\tau \\
&= (-ih\,\nabla_Y s + us)\tau + \tfrac{1}{2}chsL_Y\tau.
\end{aligned}$$

Since $L_Y\tau' = \frac{1}{2}(div_\gamma\, Y)\tau'$, we get

$$\begin{aligned}
\rho(\pi^g_{(B,c)}(\phi(Y + u))\tilde{s}) &= -ih\,\nabla_Y s|Q + us|Q + ch(div_\gamma\, Y)s|Q \\
&= \pi^k_{(L_B,c)}(Y + u)\rho\tilde{s}.
\end{aligned}$$

We have proved the following.

Theorem 4.4. *As h-representations of g_k (or g_P),*

$$\pi^g_{(B,c)} \cong \pi^k_{(L_B,c)}.$$

In particular, quantum kinematics $\pi^k_{(L_B,r+i/2)}$ is isomorphic to geometric quantization $\pi^k_{(B,r+i/2)}$ for $r \in \mathbb{R}$.

Remark. Quantum Borel r-kinematics of type 0 can be obtained from corresponding geometric quantizations where the prequantum bundles are hermitian vector bundles with fibres diffeomorphic to $\mathbb{C}^r$.

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